A heuristics approach for computing the largest eigenvalue of a pairwise comparison matrix

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Abstract

Pairwise comparison matrices (PCMs) are widely used to capture subjective human judgements, especially in the context of the Analytic Hierarchy Process (AHP). Consistency of judgements is normally computed in AHP context in the form of consistency ratio (CR), which requires estimation of the largest eigenvalue ($\lambda_{\max}$) of PCMs. Since many of these alternative methods do not require calculation of eigenvector, $\lambda_{\max}$ and hence the CR of a PCM cannot be easily estimated. We propose in this paper a simple heuristics for calculating $\lambda_{\max}$ without any need to use Eigenvector Method (EM). We illustrated the proposed procedure with larger size matrices. Simulation is used to compare the accuracy of the proposed heuristics procedure with actual $\lambda_{\max}$ for PCMs of various sizes. It has been found that the proposed heuristics is highly accurate, with errors less than 1%. The proposed procedure would avoid biases and help managers to make better decisions. The advantage of the proposed heuristics is that it can be easily calculated with simple calculations without any need for specialised mathematical procedures or software and is independent of the method used to derive priorities from PCMs.

Keywords: Multiple Criteria Analysis, Pairwise Comparison Matrix, Eigenvector Method, the Largest Eigenvalue, Consistency index.

1 Introduction

In the present information driven competitive world, multi-criteria decision making methods are becoming essential for managers and decision-makers to choose the best alternative among various alternatives that satisfies the different criteria (Stewart, 1992; Huede et. al, 2006). The usage of multi-criteria decision making started in the early 1970. Among the various techniques proposed, the Analytic Hierarchy Process (AHP) proposed by Saaty (1980) seems to be very popular and has been applied in wide variety of areas starting from planning, selecting a best alternative, resource allocations, resolving conflict, optimization, etc. (Zahedi, 1986; Vargas, 1990; Vaidya and Kumar, 2006; Hulle et al., 2013; Rahmani and Keshavarz, 2015). However, several limitations of AHP have also been reported in the literature, including rank reversal or condition of order preservation etc. (Watson and Freeling, 1982; Belton and Gear, 1983; Holder, 1990; Dyer, 1990; Salo and Hämalainen, 1997; Ramanathan and Ramanathan, 2011). Belton and Gear (1983) have reported in their note that greater attention is essential in deriving priorities and the associated scaling to enhance the initial proposed AHP method. This has stimulated an interest in alternative methods of performing the calculations required in the AHP.

The AHP proposed by Saaty (1980) typically uses the so called Eigenvector Method (EM) for deriving priorities of elements from a pairwise comparison matrix (PCM). Several methods are available to estimate priorities of elements from a PCM, the EM being the most common. Since the PCMs involve the use of human judgements, procedures to check the consistency of judgements is considered an important requirement while computing the priorities, as the priorities estimated from highly inconsistent judgements seem to be unreliable for further use. Since $\lambda_{\max}$ is automatically computed in the EM, computing CR is not a serious issue when EM is used to estimate priorities. However, EM is not the only method for estimating priorities from PCMs. Several alternatives to EM have been reported in the literature and it is reviewed in section 2. Till date a great deal of research has been carried out on alternative methods of deriving priorities from PCM in AHP. Recent study showed that simple equations or procedures for evaluation outperformed human judgment by at least 25% (Soll et al., 2015).

One of the most attractive features of AHP is its ability to estimate the consistency of comparative judgements provided by the decision maker. Suppose a_{ij} represents the elements in row i and column j of a pairwise comparison matrix denoted as A. The matrix is said to be consistent if it satisfies the following rules. $a_{ij} > 0$; $a_{ij} = \frac{1}{a_{ji}}$; $a_{ii} = 1$; and $a_{ij} = a_{ik} * a_{ki}$. Saaty (1980) has suggested that the consistency ratio (CR) of a PCM can be calculated as $CR = \frac{\left(\frac{\lambda_{\max} - n}{n-1}\right)}{RI}$, and suggested that the CR should be below 0.1 in order to accept the judgements for further calculations. The arguments are supportive in most of the applications as well as against 10% rule. Recently, Saaty and Tran (2007) have conducted a detailed analysis to ascertain that 10% rule is essential to make good decisions.

RI is called Random Index, which has been tabulated by Saaty (1980) as given below in Table 1 based on simulation experiments using randomly generated matrices of various sizes *n*. Thus, calculation of CR requires the value of the largest eigenvalue ($\lambda_{\max}$). Unfortunately, there is no easy method available for estimating $\lambda_{\max}$ for a PCM. This was not a big issue if priorities are derived using EM as $\lambda_{\max}$ is automatically calculated. However, this is an issue if priorities are calculated using alternative priority derivation methods such as the LLST (Crawford and Williams, 1985), LP based procedure (Chandran et al., 2005), CCMA (Wang et al., 2007) and DEAHP (Ramanathan, 2006). Though $\lambda_{\max}$ can be calculated in principle irrespective of the priority derivation methods employed, it requires sophisticated calculations. Golub and Vorst (2000) in their latest paper stated that numerical computation of the eigenvectors is more delicate and that leads to many challenging numerical questions on computing eigenvalues and eigenvectors in an efficient manner and accurate way. The methods for deriving weights from PCM are very simple. When the weights can be derived by simple calculations, consistency check requires complex eigenvalue calculations, and hence, it is felt that easy heuristics procedures for calculating $\lambda_{\max}$ from PCM can help greatly in estimating CR, irrespective of the choice of priority derivation methods employed.

[Insert Table 1 about Here]

This paper attempts to develop a simple heuristics procedure for calculating $\lambda_{\max}$. Overview of alternate methods is discussed in the next section. The proposed procedure is discussed in Section 3. The performance of the proposed heuristics procedure is compared using simulation in Section 4. Section 5 provides a summary and conclusions.

2 Overview of alternate methods for Eigen value computation

Brief overviews of alternate methods are as follows: Crawford (1987) proposed a geometric mean procedure based on statistical consideration (Logarithmic lease squares technique or LLST). Cogger and Yu (1985) derived new eigenweight vector method for PCM. Islei and Locket (1988) proposed a new method based on geometric least square which minimizes least square deviation and portrayed that the method can handle large data and the consistency issues. Bryson (1995) presented a goal programming method (GPM) for estimating weights of PCM. It is also highlighted that the GPM has the properties of correctness in the consistent case, comparison order invariance, smoothness and power invariance. Lipovetsky and Conklin (2002) suggested special techniques for robust estimation of priority vectors by transforming the Saaty matrix to matrix of shares of preferences and solved the eigenvaule problem for the transformed matrices. Gass and Rapcsak (2004) offered a new approach based on Singular value decomposition (SVD) for computing weights of PCM. Justified theoretically the weight derivation and compared with EM.

Laininen and Hamalainen (2003) presented formulae for evaluating the standard deviations of the estimates of the AHP-weights. Speciality of robust regression technique in terms of eliminating outliers is elaborated by comparing it with EM and LLST. Sugihara et al. (2004) proposed interval regression analysis, to incorporate decision maker’s uncertainty of judgments, which is based on the concept of possibility. Chandran et al. (2005) proposed an approach based on linear programming to estimate the weight of PCM. They incorporated interval of data in the linear programming and also performed sensitivity analysis to identify the measure of inconsistency. Wang et al. (2007) proposed a correlation coefficient maximization approach (CCMA) for estimating weights of a PCM. He proved that the CCMA can precisely estimate priorities for perfectly consistent comparison matrices and produce more than one priority estimate for inconsistent comparison matrices. Ramanathan (2006) proposed a new method combining Data Envelopment Analysis (DEA) and Analytic Hierarchy Process (AHP), called DEAHP to generate priorities from PCM. This method attempted to address the two limitations (i.e rank reversal effect and indifferent criterion flaw) pointed out by the researchers with respect to AHP.

3 A simple heuristics procedure for estimating λ_{max}

As mentioned earlier, the advantage of consistency check using the procedure suggested in Saaty (1980) is that the RI values were based on simulation and have been widely accepted. The disadvantage is that it requires estimating of λ_{max}, which is difficult to measure when priorities are calculated by methods other than EM. Hence, we propose a simple heuristics procedure to estimate λ_{max}. The heuristics procedure can be performed using simple hand calculations and is independent of priority derivation method used. Thus, the proposed heuristics procedure can be used to check the consistency of PCM when EM is used, DEAHP is used, LLSM is used or LP method is employed for deriving priorities. The formula for estimating λ_{max} for a matrix of size *n* x *n* is the following.

$$\text{Let } A = \begin{bmatrix} a_{11}=1 & a_{12} & a_{13} \\ a_{21}= 1/a_{12} & a_{22}=1 & a_{23} \\ a_{31}= 1/a_{13} & a_{32}= 1/a_{23} & a_{33}=1 \end{bmatrix}$$

$$\lambda_{max} = \frac{\left(\dfrac{a_{11}(a_{11}*a_{12}*a_{13})^{\frac{1}{3}} + a_{12}(a_{21}*a_{22}*a_{23})^{\frac{1}{3}} + a_{13}(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}{(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}}\right) + \left(\dfrac{a_{21}(a_{11}*a_{12}*a_{13})^{\frac{1}{3}} + a_{22}(a_{21}*a_{22}*a_{23})^{\frac{1}{3}} + a_{23}(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}{(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}}\right) + \left(\dfrac{a_{31}(a_{11}*a_{12}*a_{13})^{\frac{1}{3}} + a_{32}(a_{21}*a_{22}*a_{23})^{\frac{1}{3}} + a_{33}(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}{(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}\right)}{n}$$

By simplifying

$$\lambda_{max} = \frac{a_{11} + \dfrac{a_{12}(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}}{(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}} + \dfrac{a_{13}(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}{(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}} + \dfrac{a_{21}(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}}{(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}} + a_{22} + \dfrac{a_{23}(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}{(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}} + \dfrac{a_{31}(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}}{(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}} + \dfrac{a_{32}(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}}{(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}} + a_{33}}{n}$$

Substituting $a_{21} = 1/a_{12}$, $a_{31} = 1/a_{13}$, $a_{32} = 1/a_{23}$

$$\lambda_{\max} = \frac{a_{11}+a_{22}+a_{33}+\frac{a_{12}(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}}{(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}}+\frac{a_{13}(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}{(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}}+\frac{(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}}{a_{12}(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}}+\frac{a_{23}(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}{(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}}+\frac{(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}}{a_{13}(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}+\frac{(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}}{a_{23}(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}}{n}$$

Simplifying by substituting n = $a_{11}+a_{22}+a_{33}$ and geometric ratio as x_{12}, x_{13}, x_{23}

$$\lambda_{\max} = \frac{n+a_{12}x_{12}+a_{13}x_{13}+\frac{1}{a_{12}x_{12}}+a_{23}x_{23}+\frac{1}{a_{13}x_{13}}+\frac{1}{a_{23}x_{23}}}{n}$$

Where $x_{12} = \frac{(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}}{(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}}, x_{13} = \frac{(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}{(a_{11}*a_{12}*a_{13})^{\frac{1}{3}}}, x_{23} = \frac{(a_{31}*a_{32}*a_{33})^{\frac{1}{3}}}{(a_{21}*a_{22}*a_{23})^{\frac{1}{3}}}$

Therefore generalised derivation of $\lambda_{\max}$ for n x n size is

$$\lambda_{\max} = \frac{\left(n+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}a_{ij}x_{ij}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{a_{ij}x_{ij}}\right)}{n} \text{ where } x_{ij} = \left(\frac{\prod_{k=1}^{n}a_{jk}}{\prod_{k=1}^{n}a_{ik}}\right)^{\frac{1}{n}} \qquad \text{(Eq.1)}$$

The above formula is based on (1) estimating initial values of priorities for a given PCM using row geometric mean procedure of Crawford and Williams (1985), (2) estimating $\lambda_{\max}$ for each row using the eigenvector formula, and (3) averaging arithmetically the *n* values of $\lambda_{\max}$ thus obtained.

For a simple 3x3 matrix $\begin{bmatrix} 1 & a_{12} & a_{13} \\ 1/a_{12} & 1 & a_{23} \\ 1/a_{13} & 1/a_{23} & 1 \end{bmatrix}$, the above formula reduces to,

$$\lambda_{\max} = \frac{n+a_{12}x_{12}+a_{13}x_{13}+\frac{1}{a_{12}x_{12}}+a_{23}x_{23}+\frac{1}{a_{13}x_{13}}+\frac{1}{a_{23}x_{23}}}{n}$$

Where

$$x_{12} = \frac{(a_{21} * a_{22} * a_{23})^{\frac{1}{3}}}{(a_{11} * a_{12} * a_{13})^{\frac{1}{3}}}, x_{13} = \frac{(a_{31} * a_{32} * a_{33})^{\frac{1}{3}}}{(a_{11} * a_{12} * a_{13})^{\frac{1}{3}}}, x_{23} = \frac{(a_{31} * a_{32} * a_{33})^{\frac{1}{3}}}{(a_{21} * a_{22} * a_{23})^{\frac{1}{3}}}.$$

Appendix 1 illustrates calculation of $\lambda_{\max}$ and CR for a PCM of size 4.

4 Verifying the accuracy of the proposed heuristics procedure

The accuracy of (Eq.1) in correctly estimating the largest eigenvalue of a PCM is verified in this section using simulation. It may be noted that other simple formulae are available in the literature for estimating $\lambda_{\max}$. For example, Saaty (1980) has suggested the following formula for approximately calculating $\lambda_{\max}$.

$$\lambda_{\max} = \sum_{j=1}^{n} a_{ij} \frac{w_j}{W_i} \qquad \text{(Eq. 2)}$$

Where w_j represents column wise normalised weight of element a_{ij} and W_i is the average of normalised entries row wise. For a perfectly consistent matrix, both (Eq.1) and (Eq. 2) provide the correct value of $\lambda_{\max}$. However, for inconsistent matrices, the accuracy of (Eq.1) and (Eq.2) in correctly estimating $\lambda_{\max}$ varies. The accuracy also depends on the size of the matrix. We have used simulation to estimate the ability of (Eq. 1) and (Eq. 2) in correctly estimating $\lambda_{\max}$. The simulation experiment is explained below.

4.1 The simulation experiment

The simulation experiment has been carried out using Microsoft Excel. Matrices of different sizes were generated randomly using random number function of Excel. The maximum value of an element of a matrix is set to 9 to reflect the 1-9 scale of Saaty (1980). The reciprocal property ($a_{ji} = 1/a_{ij}$) and the diagonal property (a*ii* = 1) of PCM were forced. The exact value of $\lambda_{\max}$ of a PCM was estimated using the Poptools Excel Addin downloaded freely from the website of commonwealth scientific and industrial research organisation (http://www.cse.csiro.au/poptools/download.htm). This exact value is denoted as $\lambda_{\max\ \text{exact}}$ and compared with the $\lambda_{\max}$ values calculated using (Eq.1) and (Eq.2) denoted as $\lambda_{\text{max proposed}}$ and $\lambda_{\text{max Eq.2}}$ respectively. The comparison is done by calculating relative errors defined below.

For the proposed procedure (Eq.1), the per cent relative error is $\%Error_{proposed} = \frac{\lambda_{\text{max proposed}} - \lambda_{\text{max exact}}}{\lambda_{\text{max exact}}} * 100$, while for (Eq.2) the per cent relative error is $\%Error_{Eq.2} = \frac{\lambda_{\text{max Eq.2}} - \lambda_{\text{max exact}}}{\lambda_{\text{max exact}}} * 100$. The results are shown in Table 2. The significance of the proposed procedure in terms of percentage deviation is shown in last column of Table 2 to make the reader understand in a better way.

[Insert Table 2 about Here]

Thus, based on the random matrices used in the simulation experiments, the proposed heuristics procedure is able to estimate the correct value of the largest eigenvalue with errors less than 1% and accuracy more than 99%. This accuracy, coupled with the ease of calculation, makes the proposed heuristics procedure appealing to calculate λ_{max} and hence the consistency ratio of a pairwise comparison matrix.

4.2 Implications on re-engineer decision making process

Recently studies are exploring how to re-engineer decision making process that involves subjective and objective data (Davenport, 2010). The process needs simple heuristics procedure to evaluate the consistency of human judgements that are based on insufficient motivation and cognitive biases (Soll et al., 2015). Mostly manager need simple stories from a huge set of data that could use common sense methodology to make quick decisions. In a way this could prevent making expensive mistakes that are due to cognitive biases in a short span of time (Beshears and Gino, 2015). Hence our proposed procedure will be handy to check the consistency of experts' views during decision making process and to avoid expensive mistakes. Several global firms such as Google, UPS and Walmart had re-engineered their decision making process and used simple heuristics procedure to increase their profitability and customer satisfaction (Davenport, 2010).

5 Concluding remarks

A simple heuristics procedure for calculating $\lambda_{max.}$ has been proposed in this paper to facilitate verification of consistency of human judgements in pairwise comparison matrices (PCMs), typically used in the Analytic Hierarchy Process. The proposed procedure is suitable for researchers and practitioners when they employ alternative methods for deriving weights such as LLSM, LP based procedure, DEAHP, GPM, SVD, CCMA etc. The proposed heuristics procedure is very simple and can be easily performed using hand calculations. Calculations using the proposed heuristics procedure has been illustrated for a PCM of size 4 in the Appendix. The accuracy of the proposed procedure is verified through simulation and it was found that the proposed procedure is more than 99% accurate in estimating the correct value of λ_{max}.

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APPENDIX 1 ILLUSTRATION OF THE HEURISTIC PROCEDURE

This section illustrates the simple heuristics procedure used to derive the largest eigenvalue and consistency index with a 4 x 4 and 8x8 size matrices.

<u>4x4 matrix illustration</u>

Let $A = \begin{matrix} 1 & 2 & 2 & 8 \\ 0.5 & 1 & 1 & 4 \\ 0.5 & 1 & 1 & 6 \\ 0.125 & 0.25 & 0.166667 & 1 \end{matrix}$

$$\lambda_{\max} = \frac{\left(n + a_{12}x_{12} + a_{13}x_{13} + a_{14}x_{14} + a_{23}x_{23} + a_{24}x_{24} + a_{34}x_{34} + \left(\frac{1}{a_{12}x_{12}} + \frac{1}{a_{13}x_{13}} + \frac{1}{a_{14}x_{14}} + \frac{1}{a_{23}x_{23}} + \frac{1}{a_{24}x_{24}} + \frac{1}{a_{34}x_{34}}\right)\right)}{n}$$

$$x_{12} = \frac{(a_{21} * a_{22} * a_{23} * a_{24})^{\frac{1}{4}}}{(a_{11} * a_{12} * a_{13} * a_{14})^{\frac{1}{4}}} = \frac{(0.5 * 1 * 1 * 4)^{\frac{1}{4}}}{(1 * 2 * 2 * 8)^{\frac{1}{4}}} = \frac{1.189207}{2.378414} = 0.5$$

$$x_{13} = \frac{(a_{31} * a_{32} * a_{33} * a_{34})^{\frac{1}{4}}}{(a_{11} * a_{12} * a_{13} * a_{14})^{\frac{1}{4}}} = \frac{(0.5 * 1 * 1 * 6)^{\frac{1}{4}}}{(1 * 2 * 2 * 8)^{\frac{1}{4}}} = \frac{1.316074}{2.378414} = 0.55334$$

$$x_{14} = \frac{(a_{41} * a_{42} * a_{43} * a_{44})^{\frac{1}{4}}}{(a_{11} * a_{12} * a_{13} * a_{14})^{\frac{1}{4}}} = \frac{(0.125 * 0.25 * 0.166667 * 1)^{\frac{1}{4}}}{(1 * 2 * 2 * 8)^{\frac{1}{4}}} = \frac{0.268642}{2.378414} = 0.112950$$

$$x_{23} = \frac{(a_{31} * a_{32} * a_{33} * a_{34})^{\frac{1}{4}}}{(a_{21} * a_{22} * a_{23} * a_{24})^{\frac{1}{4}}} = \frac{(0.5 * 1 * 1 * 6)^{\frac{1}{4}}}{(0.5 * 1 * 1 * 4)^{\frac{1}{4}}} = \frac{1.316074}{1.189207} = 1.106682$$

$$x_{24} = \frac{(a_{41} * a_{42} * a_{43} * a_{44})^{\frac{1}{4}}}{(a_{21} * a_{22} * a_{23} * a_{24})^{\frac{1}{4}}} = \frac{(0.125 * 0.25 * 0.166667 * 1)^{\frac{1}{4}}}{(0.5 * 1 * 1 * 4)^{\frac{1}{4}}} = \frac{0.268642}{1.189207} = 0.225900$$

$$x_{34} = \frac{(a_{41} * a_{42} * a_{43} * a_{44})^{\frac{1}{4}}}{(a_{31} * a_{32} * a_{33} * a_{34})^{\frac{1}{4}}} = \frac{(0.125 * 0.25 * 0.166667 * 1)^{\frac{1}{4}}}{(0.5 * 1 * 1 * 6)^{\frac{1}{4}}} = \frac{0.268642}{1.316074} = 0.204123$$

$$\lambda_{\max} = \frac{\left(\begin{matrix} 4 + 2(0.5) + 2(0.55334) + 8(0.112950) + 1(1.106682) + 4(0.2259) + 6(0.204123) + \\ \left(\frac{1}{2(0.5)} + \frac{1}{2(0.55334)} + \frac{1}{8(0.112950)} + \frac{1}{1(1.106682)} + \frac{1}{4(0.2259)} + \frac{1}{6(0.204123)}\right) \end{matrix}\right)}{4}$$

$\lambda_{\max} = 4.2625$, Consistency Index (CI) = 0.09722

<u>8X8 mtarix illustration</u>

$$\text{Let } B = \begin{matrix} 1 & 2 & 1 & 3 & 5 & 5 & 3 & 3 \\ 0.5 & 1 & 2 & 1 & 2 & 1 & 7 & 4 \\ 1 & 0.5 & 1 & 1 & 3 & 6 & 6 & 2 \\ 0.33 & 1 & 1 & 1 & 7 & 6 & 4 & 4 \\ 0.2 & 0.5 & 0.33 & 0.14 & 1 & 1 & 4 & 2 \\ 0.2 & 1 & 0.16 & 0.16 & 1 & 1 & 2 & 1 \\ 0.33 & 0.14 & 0.16 & 0.25 & 0.25 & 0.5 & 1 & 1 \\ 0.33 & 0.25 & 0.5 & 0.25 & 0.5 & 1 & 1 & 1 \end{matrix}$$

$$\lambda_{\max} = \frac{\left(n + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a_{ij} x_{ij} + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{1}{a_{ij} x_{ij}} \right)}{n} = \frac{\left(8 + \sum_{i=1}^{7} \sum_{j=i+1}^{8} a_{ij} x_{ij} + \sum_{i=1}^{7} \sum_{j=i+1}^{8} \frac{1}{a_{ij} x_{ij}} \right)}{8}$$

$$x_{12} = \frac{(a_{21} * a_{22} * a_{23} * a_{24} * a_{25} * a_{26} * a_{27} * a_{28})^{\frac{1}{8}}}{(a_{11} * a_{12} * a_{13} * a_{14} * a_{15} * a_{16} * a_{17} * a_{18})^{\frac{1}{8}}} = \frac{(0.5*1*2*1*2*1*7*4)^{1/8}}{(1*2*1*3*5*5*3*3)^{1/8}} = \frac{1.65}{2.46} = 0.67$$

$$x_{13} = \frac{(a_{31} * a_{32} * a_{33} * a_{34} * a_{35} * a_{36} * a_{37} * a_{38})^{\frac{1}{8}}}{(a_{11} * a_{12} * a_{13} * a_{14} * a_{15} * a_{16} * a_{17} * a_{18})^{\frac{1}{8}}} = \frac{(1*0.5*1*1*3*6*6*2)^{1/8}}{(1*2*1*3*5*5*3*3)^{1/8}} = \frac{1.79}{2.46} = 0.73$$

$$x_{14} = \frac{(a_{41} * a_{42} * a_{43} * a_{44} * a_{45} * a_{46} * a_{47} * a_{48})^{\frac{1}{8}}}{(a_{11} * a_{12} * a_{13} * a_{14} * a_{15} * a_{16} * a_{17} * a_{18})^{\frac{1}{8}}} = \frac{(0.33*1*1*1*7*6*4*4)^{1/8}}{(1*2*1*3*5*5*3*3)^{1/8}} = \frac{1.96}{2.46} = 0.80$$

$$x_{15} = \frac{(a_{51} * a_{52} * a_{53} * a_{54} * a_{55} * a_{56} * a_{57} * a_{58})^{\frac{1}{8}}}{(a_{11} * a_{12} * a_{13} * a_{14} * a_{15} * a_{16} * a_{17} * a_{18})^{\frac{1}{8}}} = \frac{(0.2*0.5*0.33*0.14*1*1*4*2)^{1/8}}{(1*2*1*3*5*5*3*3)^{1/8}} = \frac{0.66}{2.46} = 0.27$$

$$x_{16} = \frac{(a_{61} * a_{62} * a_{63} * a_{64} * a_{65} * a_{66} * a_{67} * a_{68})^{\frac{1}{8}}}{(a_{11} * a_{12} * a_{13} * a_{14} * a_{15} * a_{16} * a_{17} * a_{18})^{\frac{1}{8}}} = \frac{(0.2*1*0.16*0.16*1*1*2*1)^{1/8}}{(1*2*1*3*5*5*3*3)^{1/8}} = \frac{0.56}{2.46} = 0.23$$

$$x_{17} = \frac{(a_{71} * a_{72} * a_{73} * a_{74} * a_{75} * a_{76} * a_{77} * a_{78})^{\frac{1}{8}}}{(a_{11} * a_{12} * a_{13} * a_{14} * a_{15} * a_{16} * a_{17} * a_{18})^{\frac{1}{8}}} = \frac{(0.33*0.14*0.16*0.25*0.25*0.5*1*1)^{1/8} =}{(1*2*1*3*5*5*3*3)^{1/8}} = \frac{0.35}{2.46} = 0.14$$

$$x_{18}=\frac{(a_{81}*a_{82}*a_{83}*a_{84}*a_{85}*a_{86}*a_{87}*a_{88})^{\frac{1}{8}}}{(a_{11}*a_{12}*a_{13}*a_{14}*a_{15}*a_{16}*a_{17}*a_{18})^{\frac{1}{8}}}=\frac{(0.33*0.25*0.5*0.25*0.5*1*1*1)^{1/8}=}{(1*2*1*3*5*5*3*3)^{1/8}}=\frac{0.52}{2.46}=0.21$$

$$x_{23}=\frac{(a_{31}*a_{32}*a_{33}*a_{34}*a_{35}*a_{36}*a_{37}*a_{38})^{\frac{1}{8}}}{(a_{21}*a_{22}*a_{23}*a_{24}*a_{25}*a_{26}*a_{27}*a_{28})^{\frac{1}{8}}}=\frac{(1*0.5*1*1*3*6*6*2)^{1/8}}{(0.5*1*2*1*2*1*7*4)^{1/8}}=\frac{1.79}{1.65}=1.09$$

$$x_{24}=\frac{(a_{41}*a_{42}*a_{43}*a_{44}*a_{45}*a_{46}*a_{47}*a_{48})^{\frac{1}{8}}}{(a_{21}*a_{22}*a_{23}*a_{24}*a_{25}*a_{26}*a_{27}*a_{28})^{\frac{1}{8}}}=\frac{(0.33*1*1*1*7*6*4*4)^{1/8}}{(0.5*1*2*1*2*1*7*4)^{1/8}}=\frac{1.96}{1.65}=1.19$$

$$x_{25}=\frac{(a_{51}*a_{52}*a_{53}*a_{54}*a_{55}*a_{56}*a_{57}*a_{58})^{\frac{1}{8}}}{(a_{21}*a_{22}*a_{23}*a_{24}*a_{25}*a_{26}*a_{27}*a_{28})^{\frac{1}{8}}}=\frac{(0.2*0.5*0.33*0.14*1*1*4*2)^{1/8}}{(0.5*1*2*1*2*1*7*4)^{1/8}}=\frac{0.66}{1.65}=0.40$$

$$x_{26}=\frac{(a_{61}*a_{62}*a_{63}*a_{64}*a_{65}*a_{66}*a_{67}*a_{68})^{\frac{1}{8}}}{(a_{21}*a_{22}*a_{23}*a_{24}*a_{25}*a_{26}*a_{27}*a_{28})^{\frac{1}{8}}}=\frac{(0.2*1*0.16*0.16*1*1*2*1)^{1/8}}{(0.5*1*2*1*2*1*7*4)^{1/8}}=\frac{0.56}{1.65}=0.34$$

$$x_{27}=\frac{(a_{71}*a_{72}*a_{73}*a_{74}*a_{75}*a_{76}*a_{77}*a_{78})^{\frac{1}{8}}}{(a_{21}*a_{22}*a_{23}*a_{24}*a_{25}*a_{26}*a_{27}*a_{28})^{\frac{1}{8}}}=\frac{(0.33*0.14*0.16*0.25*0.25*0.5*1*1)^{1/8}}{(0.5*1*2*1*2*1*7*4)^{1/8}}=\frac{0.35}{1.65}=0.21$$

$$x_{28}=\frac{(a_{81}*a_{82}*a_{83}*a_{84}*a_{85}*a_{86}*a_{87}*a_{88})^{\frac{1}{8}}}{(a_{21}*a_{22}*a_{23}*a_{24}*a_{25}*a_{26}*a_{27}*a_{28})^{\frac{1}{8}}}=\frac{(0.33*0.25*0.5*0.25*0.5*1*1*1)^{1/8}}{(0.5*1*2*1*2*1*7*4)^{1/8}}=\frac{0.52}{1.65}=0.31$$

$$x_{34}=\frac{(a_{41}*a_{42}*a_{43}*a_{44}*a_{45}*a_{46}*a_{47}*a_{48})^{\frac{1}{8}}}{(a_{31}*a_{32}*a_{33}*a_{34}*a_{35}*a_{36}*a_{37}*a_{38})^{\frac{1}{8}}}=\frac{(0.33*1*1*1*7*6*4*4)^{1/8}}{(1*0.5*1*1*3*6*6*2)^{1/8}}=\frac{1.96}{1.79}=1.09$$

$$x_{35}=\frac{(a_{51}*a_{52}*a_{53}*a_{54}*a_{55}*a_{56}*a_{57}*a_{58})^{\frac{1}{8}}}{(a_{31}*a_{32}*a_{33}*a_{34}*a_{35}*a_{36}*a_{37}*a_{38})^{\frac{1}{8}}}=\frac{(0.2*0.5*0.33*0.14*1*1*4*2)^{1/8}}{(1*0.5*1*1*3*6*6*2)^{1/8}}=\frac{0.66}{1.79}=0.37$$

$$x_{36}=\frac{(a_{61}*a_{62}*a_{63}*a_{64}*a_{65}*a_{66}*a_{67}*a_{68})^{\frac{1}{8}}}{(a_{31}*a_{32}*a_{33}*a_{34}*a_{35}*a_{36}*a_{37}*a_{38})^{\frac{1}{8}}}=\frac{(0.2*1*0.16*0.16*1*1*2*1)^{1/8}}{(1*0.5*1*1*3*6*6*2)^{1/8}}=\frac{0.56}{1.79}=0.31$$

$$x_{37} = \frac{(a_{71} * a_{72} * a_{73} * a_{74} * a_{75} * a_{76} * a_{77} * a_{78})^{\frac{1}{8}}}{(a_{31} * a_{32} * a_{33} * a_{34} * a_{35} * a_{36} * a_{37} * a_{38})^{\frac{1}{8}}} = \frac{(0.33*0.14*0.16*0.25*0.25*0.5*1*1)^{1/8}}{(1*0.5*1*1*3*6*6*2)^{1/8}} = \frac{0.35}{1.79} = 0.20$$

$$x_{38} = \frac{(a_{81} * a_{82} * a_{83} * a_{84} * a_{85} * a_{86} * a_{87} * a_{88})^{\frac{1}{8}}}{(a_{31} * a_{32} * a_{33} * a_{34} * a_{35} * a_{36} * a_{37} * a_{38})^{\frac{1}{8}}} = \frac{(0.33*0.25*0.5*0.25*0.5*1*1*1)^{1/8}}{(1*0.5*1*1*3*6*6*2)^{1/8}} = \frac{0.52}{1.79} = 0.55$$

$$x_{45} = \frac{(a_{51} * a_{52} * a_{53} * a_{54} * a_{55} * a_{56} * a_{57} * a_{58})^{\frac{1}{8}}}{(a_{41} * a_{42} * a_{43} * a_{44} * a_{45} * a_{46} * a_{47} * a_{48})^{\frac{1}{8}}} = \frac{(0.2*0.5*0.33*0.14*1*1*4*2)^{1/8}}{(0.33*1*1*1*7*6*4*4)^{1/8}} = \frac{0.66}{1.96} = 0.34$$

$$x_{46} = \frac{(a_{61} * a_{62} * a_{63} * a_{64} * a_{65} * a_{66} * a_{67} * a_{68})^{\frac{1}{8}}}{(a_{41} * a_{42} * a_{43} * a_{44} * a_{45} * a_{46} * a_{47} * a_{48})^{\frac{1}{8}}} = \frac{(0.2*1*0.16*0.16*1*1*2*1)^{1/8}}{(0.33*1*1*1*7*6*4*4)^{1/8}} = \frac{0.56}{1.96} = 0.29$$

$$x_{47} = \frac{(a_{71} * a_{72} * a_{73} * a_{74} * a_{75} * a_{76} * a_{77} * a_{78})^{\frac{1}{8}}}{(a_{41} * a_{42} * a_{43} * a_{44} * a_{45} * a_{46} * a_{47} * a_{48})^{\frac{1}{8}}} = \frac{(0.33*0.14*0.16*0.25*0.25*0.5*1*1)^{1/8}}{(0.33*1*1*1*7*6*4*4)^{1/8}} = \frac{0.35}{1.96} = 0.18$$

$$x_{48} = \frac{(a_{81} * a_{82} * a_{83} * a_{84} * a_{85} * a_{86} * a_{87} * a_{88})^{\frac{1}{8}}}{(a_{41} * a_{42} * a_{43} * a_{44} * a_{45} * a_{46} * a_{47} * a_{48})^{\frac{1}{8}}} = \frac{(0.33*0.25*0.5*0.25*0.5*1*1*1)^{1/8}}{(0.33*1*1*1*7*6*4*4)^{1/8}} = \frac{0.52}{1.96} = 0.26$$

$$x_{56} = \frac{(a_{61} * a_{62} * a_{63} * a_{64} * a_{65} * a_{66} * a_{67} * a_{68})^{\frac{1}{8}}}{(a_{51} * a_{52} * a_{53} * a_{54} * a_{55} * a_{56} * a_{57} * a_{58})^{\frac{1}{8}}} = \frac{(0.2*1*0.16*0.16*1*1*2*1)^{1/8}}{(0.2*0.5*0.33*0.14*1*1*4*2)^{1/8}} = \frac{0.56}{0.66} = 0.85$$

$$x_{57} = \frac{(a_{71} * a_{72} * a_{73} * a_{74} * a_{75} * a_{76} * a_{77} * a_{78})^{\frac{1}{8}}}{(a_{51} * a_{52} * a_{53} * a_{54} * a_{55} * a_{56} * a_{57} * a_{58})^{\frac{1}{8}}} = \frac{(0.33*0.14*0.16*0.25*0.25*0.5*1*1)^{1/8}}{(0.2*0.5*0.33*0.14*1*1*4*2)^{1/8}} = \frac{0.35}{0.66} = 0.53$$

$$x_{58} = \frac{(a_{81} * a_{82} * a_{83} * a_{84} * a_{85} * a_{86} * a_{87} * a_{88})^{\frac{1}{8}}}{(a_{51} * a_{52} * a_{53} * a_{54} * a_{55} * a_{56} * a_{57} * a_{58})^{\frac{1}{8}}} = \frac{(0.33*0.25*0.5*0.25*0.5*1*1*1)^{1/8}}{(0.2*0.5*0.33*0.14*1*1*4*2)^{1/8}} = \frac{0.52}{0.66} = 0.78$$

$$x_{67} = \frac{(a_{71} * a_{72} * a_{73} * a_{74} * a_{75} * a_{76} * a_{77} * a_{78})^{\frac{1}{8}}}{(a_{61} * a_{62} * a_{63} * a_{64} * a_{65} * a_{66} * a_{67} * a_{68})^{\frac{1}{8}}} = \frac{(0.33*0.14*0.16*0.25*0.25*0.5*1*1)^{1/8}}{(0.2*1*0.16*0.16*1*1*2*1)^{1/8}} = \frac{0.35}{0.56} = 0.62$$

$$x_{68} = \frac{(a_{81} * a_{82} * a_{83} * a_{84} * a_{85} * a_{86} * a_{87} * a_{88})^{\frac{1}{8}}}{(a_{61} * a_{62} * a_{63} * a_{64} * a_{65} * a_{66} * a_{67} * a_{68})^{\frac{1}{8}}} = \frac{(0.33*0.25*0.5*0.25*0.5*1*1*1)^{1/8}}{(0.2*1*0.16*0.16*1*1*2*1)^{1/8}} = \frac{0.52}{0.56} = 0.92$$

$$x_{78}=\frac{(a_{81}*a_{82}*a_{83}*a_{84}*a_{85}*a_{86}*a_{87}*a_{88})^{\frac{1}{8}}}{(a_{71}*a_{72}*a_{73}*a_{74}*a_{75}*a_{76}*a_{77}*a_{78})^{\frac{1}{8}}}=\frac{(0.33*0.25*0.5*0.25*0.5*1*1*1)^{1/8}}{(0.33*0.14*0.16*0.25*0.25*0.5*1*1)^{1/8}}=\frac{0.52}{0.35}=1.47$$

$$\lambda_{\max}=\frac{\left(n+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}a_{ij}x_{ij}+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{a_{ij}x_{ij}}\right)}{n}=\frac{\left(8+\sum_{i=1}^{7}\sum_{j=i+1}^{8}a_{ij}x_{ij}+\sum_{i=1}^{7}\sum_{j=i+1}^{8}\frac{1}{a_{ij}x_{ij}}\right)}{8}$$

$\lambda_{\max}=8.872936$, Consistency Index (CI) = 0.088443318

Table 1 Random index (Source: Saaty, 1980)

N	3	4	5	6	7	8	9
RI	0.58	0.90	1.12	1.24	1.32	1.41	1.45

Table 2 Comparison of the proposed heuristics procedure and Eq. 2 in estimating exact value of $\lambda_{\max}$

Size of matrices	Consistency Ratio (CR)	No of random matrices	%Error$_{\text{proposed}}$ Min	Max	Average	%Error$_{\text{Eq.2}}$ Min	Max	Average	Improvement of proposed procedure over Eq. 2 $\frac{\text{AverageError}_{\text{Eq.2}}-\text{AverageError}_{\text{proposed}}}{\text{AverageError}_{\text{proposed}}}*100$
9x9	<0.1	1001	0.000	0.008	0.23%	0.000	0.040	1.1%	374 %
	0.1<CR<0.2	1001	0.000	0.017	0.59%	0.000	0.084	1.96%	231
	0.2<CR<0.3	1001	0.000	0.027	0.96%	0.000	0.092	2.39%	149
	0.3<CR<0.4	1001	0.000	0.027	0.94%	0.000	0.108	2.79%	197
	0.4<CR<0.5	1001	0.000	0.026	0.85%	0.000	0.119	2.74%	223
8x8	<0.1	1001	0.000	0.006	0.21%	0.000	0.047	1.15%	448
	0.1<CR<0.2	1001	0.000	0.019	0.55%	0.000	0.096	2.01%	266
	0.2<C.R<0.3	1001	0.000	0.028	0.95%	0.000	0.102	2.72%	187
	0.3<CR<0.4	1001	0.000	0.021	0.77%	0.000	0.108	3.21%	317
	0.4<CR<0.5	1001	0.000	0.023	0.81%	0.000	0.120	3.34%	313
7x7	< 0.1	1001	0.000	0.006	0.16%	0.000	0.048	0.94%	488
	0.1<CR<0.2	1001	0.000	0.020	0.47%	0.000	0.070	1.79%	281
	0.2<CR<0.3	1001	0.000	0.026	0.94%	0.000	0.116	2.59%	176
	0.3<CR<0.4	1001	0.000	0.026	0.91%	0.000	0.120	3.11%	242
	0.4<CR<0.5	1001	0.000	0.030	0.96%	0.000	0.144	2.93%	206
6x6	< 0.1	1001	0.000	0.005	0.12%	0.000	0.044	0.93%	675
	0.1<CR<0.2	1001	0.000	0.015	0.41%	0.000	0.082	1.88%	359
	0.2<CR<0.3	1001	0.000	0.026	0.79%	0.000	0.128	2.77%	251
	0.3<CR<0.4	1001	0.000	0.026	0.76%	0.000	0.160	3.54%	366
	0.4<CR<0.5	1001	0.000	0.024	0.84%	0.000	0.151	4.04%	381
5x5	<0.1	1001	0.000	0.001	0.05%	0.000	0.021	0.45%	800
	0.1<CR<0.2	1001	0.001	0.005	0.27%	0.000	0.042	1.18%	338
	0.2<C.R<0.3	1001	0.003	0.008	0.59%	0.000	0.058	1.65%	180
	0.3<C.R<0.4	1001	0.000	0.023	0.64%	0.000	0.135	3.69%	477
	0.4<C.R<0.5	1001	0.000	0.025	0.83%	0.000	0.155	4.24%	411
4x4	<0.1	1001	0.000	0.000	0.02%	0.000	0.022	0.43%	2050
	0.1<CR<0.2	1001	0.000	0.002	0.15%	0.000	0.043	2.07%	1280
	0.2<C.R<0.3	1001	0.002	0.003	0.28%	0.000	0.061	3.12%	1015
	0.3<C.R<0.4	1001	0.000	0.012	0.38%	0.000	0.106	3.26%	758
	0.4<C.R<0.5	1001	0.000	0.017	0.50%	0.000	0.138	3.83%	666
3x3	<0.1	1001	0	0	0	0	0	0	0
	0.1<CR<0.2	1001	0	0	0	0	0	0	0
	0.2<CR<0.3	1001	0	0	0	0	0	0	0
	0.3<C.R<0.4	1001	0	0	0	0	0	0	0
	0.4<C.R<0.5	1001	0	0	0	0	0	0	0